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Scaling Factors Associated with M-Furcations of the $1-\mu\,|x|^z$ Map

M. C. de Sousa Vieira[1]

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A numerical study is made of the scaling behavior associated with M-furcations ($M = 3, 4, 5$) in the map $x_{t+1} = 1 - \mu\,|x_t|^z$ $(z > 1)$. The scaling constants δ and α are calculated as functions of z, as well as the more general scaling functions σ and $f(a)$.

KEY WORDS: Chaos; one-dimensional maps; multifractality.

1. INTRODUCTION

The one-dimensional iterative map

$$x_{t+1} = f(x_t) \equiv 1 - \mu\,|x_t|^z, \qquad z > 1 \tag{1}$$

which maps the interval $x \in [-1, 1]$ into itself, displays a very rich dynamical behavior.[1,2] This map is generic for all single-hump one-dimensional maps which have (locally around the maximum) a leading nonlinearity of order z. The $z = 2$ case is by far the most common in experiments,[3] but other values of z are also found.[4]

When the parameter μ in Eq. (1) is raised (starting from $\mu = 0$) the attractors (or long-time solutions) of the map show a sequence of periodic orbits with period 2^k $(k = 0, 1, 2...)$. The kth period appears at μ_k through a pitchfork bifurcation of the $(k-1)$th period, and the sequence $\{\mu_k\}$ accumulates $(k \to \infty)$ at $\mu_\infty(z)$, where the system enters into chaos. In the chaotic region aperiodic attractors are present as well as an infinite number of periodic windows, which always appear in the same order, independently of z. When these windows are taken in an appropriate order, they

[1] Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud, 150, 22290-Rio de Janeiro, RJ, Brazil.

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form sequences of M-furcations, with period M^k ($M > 2$), which generalize the bifurcations ($M = 2$). The windows of the M-furcations are not adjacent on the parameter axis and they are very narrow. However, the period-tripling sequence ($M = 3$) has been observed experimentally.[5] Above $\mu = 2$ no finite attractors exist and x_t is driven to infinity.

All the sequences of M-furcations present scaling factors that converge and define universality classes determined by z, in the sense that the factors do not change if higher-order terms are included in Eq. (1). In the μ direction there is the scaling factor δ and in the x direction there is a whole set of scaling factors (the principal ones being α and α^z) which together form the scaling function σ. The existence of a set of scaling indices in the attractor at the accumulation point of the M-furcation characterizes the presence of a multifractal, which can be studied through the function $f(a)$.

Van der Weele *et al.*[6] studied δ, α, σ, and $f(a)$ as function of z for the bifurcations ($M = 2$), and Shau-Jin Chang *et al.*[7] calculated α, δ, and fractal dimensions for the $z = 2$ case and $M \leqslant 7$. References 8–14 also deal with scaling factors for the M-furcations in the map (1).

The aim of the present communication is to study numerically, as function of z, the scaling factors α and δ, and the scaling functions σ and $f(a)$ for $M = 3, 4$, and 5, which correspond, respectively, to trifurcations, tetrafurcations, and pentafurcations. The paper is organized as follows: in the next section I calculate the scaling function δ; Sections II and III study the functions σ and $f(a)$, respectively; the last section is dedicated to the conclusions.

2. THE SCALING FACTOR δ

In this section let us initially fix upon notations before introducing the method used in the numerical calculations. For every periodic orbit in the map (1) there is one value of the control parameter for which the orbit includes the critical point (peak) of the map. At this value of the parameter the cycle is called superstable. Following the images of the peak at the superstable cycle it is possible to form a word of R and L according to whether the subsequent iterations in the orbit are on the right or on the left of the peak. This word is called the U-sequence of the cycle.[15] In the case of the trifurcations and tetrafurcations the basic 3-cycle and 4-cycle have U-sequences RL and RLL (or RL^2). The pentafurcations have three types of sequences for the basic 5-cycle, namely RLR^2, $\mathrm{RL}^2\mathrm{R}$, and RL^3. The U-sequences related to the higher-order periods in the M-furcations are constructed following the rules described in refs. 1 and 15.

For each family of cycles related to a sequence of M-furcations the set

$\{\tilde{\mu}_k\}$ where superstable cycles occur converges geometrically at a rate given by

$$\delta = \lim_{k\to\infty} \frac{\tilde{\mu}_k - \tilde{\mu}_{k-1}}{\tilde{\mu}_{k+1} - \tilde{\mu}_k} = \text{const} \tag{2}$$

and it accumulates at $\tilde{\mu}_\infty$.

To determine the set $\{\tilde{\mu}_k\}$ where the cycles belonging to a family of M-furcations are superstable, we use the method introduced by Hao Bai-Lin.(16) Let us explain the method with the same type of U-sequence chosen by him, namely RLRR. At the superstable cycle the iterations of the map proceed from $x=0$ to $x=0$, i.e.,

$$f_{\mathrm{R}}(f_{\mathrm{R}}(f_{\mathrm{L}}(f_{\mathrm{R}}(f(\mu, 0))))) = 0 \tag{3}$$

where the subscript R or L indicates which branch, right or left, of the map has been used at each iteration. Since the inverse mapping is two valued, define

$$R(x) = f_{\mathrm{R}}^{-1}(\mu, x) = +[(1-x)/\mu]^{1/z} \tag{4a}$$

$$L(x) = f_{\mathrm{L}}^{-1}(\mu, x) = -[(1-x)/\mu]^{1/z} \tag{4b}$$

depending on which half of the mapping is used. If successive inverses of Eq. (3) are taken, one obtains for the word RLRR the functional relation

$$R(L(R(R(0)))) = 1 \tag{5}$$

which is an equation for μ. Multiplying Eq. (5) by $\beta \equiv 1/\mu$ yields

$$\beta[\beta(1 + [\beta(1 - [\beta(1 - \beta^{1/z})]^{1/z})]^{1/z})]^{1/z} = \beta \tag{6}$$

This equation can be solved by iterations, as suggested by Kaplan,(17) i.e., replacing it by

$$\beta_{n+1} = \beta_n[\beta_n(1 + [\beta_n(1 - [\beta_n(1 - \beta_n^{1/z})]^{1/z})]^{1/z})]^{1/z} \tag{7}$$

and then iterating the equation for a suitable β_0.

This method is quite simple and can be used for any type of map whose inverse is calculable in a closed form and for any type of U-sequence. However, the convergence of Eq. (7) becomes slower in the $z\to 1$ and $z\to\infty$ limits.

Table I displays the values of the accumulation points $\tilde{\mu}_\infty$ of the superstable values and of δ for $z = 1.5$, 2, 3, 4, 6, 8, and 10 and for five types of sequences, namely $(\mathrm{RL})^{*n}$, $(\mathrm{RL}^2)^{*n}$, $(\mathrm{RLR}^2)^{*n}$, $(\mathrm{RL}^2\mathrm{R})^{*n}$, and

Table I. Accumulation Points $\tilde{\mu}_\infty$ and M-Furcation Rates δ and α for Typical Values of z and for the Sequences $(RL)^{*n}$, $(RL^2)^{*n}$, $(RLR^2)^{*n}$, $(RL^2R)^{*n}$, and $(RL^3)^{*n}$ [a]

z		$(RL)^{*n}$	$(RL^2)^{*n}$	$(RLR^2)^{*n}$	$(RL^2R)^{*n}$	$(RL^3)^{*n}$
1.5	$\tilde{\mu}_\infty$	1.713540707	1.908140938	1.581073957	1.810146096	1.970391709
	δ	7.311×10^1	2.719×10^3	6.442×10^2	4.984×10^3	8.691×10^4
	α	3.010×10^1	3.234×10^2	1.292×10^2	5.143×10^2	3.174×10^3
2	$\tilde{\mu}_\infty$	1.786440255	1.942704354	1.631926654	1.862224022	1.985539530
	δ	5.524×10^1	9.816×10^2	2.555×10^2	1.287×10^3	1.693×10^4
	α	9.277	3.882×10^1	2.013×10^1	4.580×10^1	1.600×10^2
3	$\tilde{\mu}_\infty$	1.867865948	1.973456485	1.700204726	1.918298028	1.995250019
	δ	6.681×10^1	9.665×10^2	2.404×10^2	1.106×10^3	1.486×10^4
	α	4.364	1.063×10^1	6.720	1.125×10^1	2.645×10^1
4	$\tilde{\mu}_\infty$	1.909335470	1.985504660	1.743351015	1.945858583	1.997974021
	δ	8.578×10^1	1.275×10^3	2.919×10^2	1.418×10^3	2.099×10^4
	α	3.152	6.193	4.294	6.398	1.248×10^1
6	$\tilde{\mu}_\infty$	1.948866269	1.994205417	1.795920044	1.970972615	1.999432441
	δ	1.301×10^2	2.22×10^3	4.317×10^2	2.433×10^3	4.32×10^4
	α	2.281	3.659	2.790	3.727	6.007
8	$\tilde{\mu}_\infty$	1.966776434	1.997084404	1.827674871	1.981779236	1.999779411
	δ	1.789×10^2	3.49×10^3	5.89×10^2	3.79×10^3	7.87×10^4
	α	1.925	2.791	2.237	2.826	4.122
10	$\tilde{\mu}_\infty$	1.976500608	1.998317004	1.849408804	1.987431027	1.999896134
	δ	2.296×10^2	5.05×10^3	7.53×10^2	5.42×10^3	1.28×10^5
	α	1.729	2.355	1.949	2.375	3.26

[a] The results for $z=2$, 4, 6, and 8 are also calculated in ref. 10, but the present numerical values are more accurate.

$(RL^3)^{*n}$ (see ref. 1 for this notation), which correspond to $M=3$, 4, and 5. The numerical values of δ as a function of z are plotted in Fig. 1. Note that δ seems to diverge in the limit $z\to1$ for all cases considered and presents a minimum near $z=2$. For the bifurcations ($M=2$), $\delta(z)$ is a monotonically increasing function of z and $\delta(1)=2$.[1] There is a controversy about the behavior of $\lim_{z\to\infty}\delta(z)$. Eckmann and Wittwer[11] and van der Weele *et al.*[6,8] affirm that $\lim_{z\to\infty}\delta(t)\lesssim30$, whereas Bhattachargee and Banerjee[13] claim that $\delta(z)$ diverges in the $z\to\infty$ limit (the latter result coincides with the one suggested via the renormalization group[12]). For $M>2$ I do not know any conjecture about the behavior of $\delta(z)$ in the $z\to\infty$ limit. Numerical calculations in this limit are very difficult, since the convergence of δ becomes very slow.

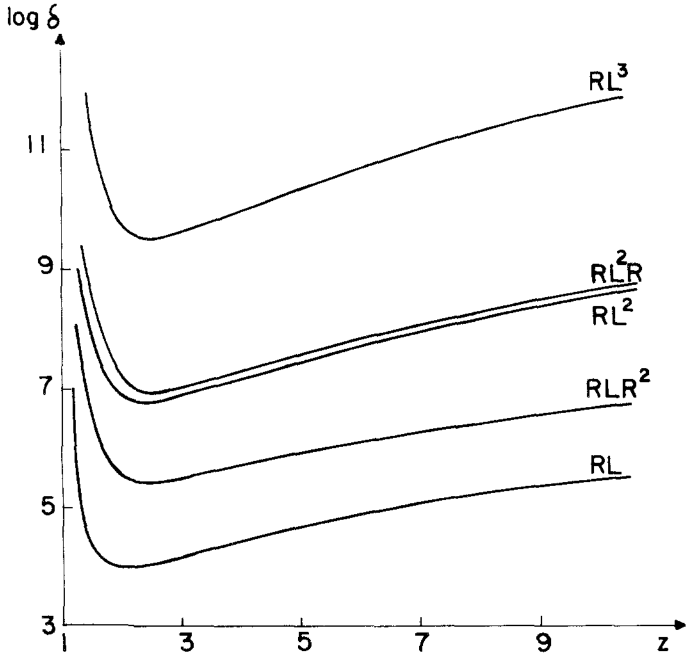

Fig. 1. Log plot of the scaling factor δ for the U-sequences with $M=3$, 4, and 5.

3. THE FUNCTION σ

In the x direction there is a whole set of scaling indices associated with the attractors at the accumulation point of the M-furcations; this fact characterizes the presence of a multifractal. The principal indices are α and α^z, which are related to central (near $x=0$) and top (near $x=1$) distances of the M-furcation tree, respectively. To determine the function σ, consider the superstable M^k-cycle $\{x_0, x_1,..., x_{M^k-1}\}$ with $x_0=0$. The distances between x_m and $x_{m+M^{k-1}}$ $(0<m\leqslant M^{k-1})$ are given by

$$\begin{aligned} d_{k,m} &= |x_{m+M^{k-1}} - x_m| \\ &= |f_{\bar{\mu}_k}^{(m+M^{k-1})}(0) - f_{\bar{\mu}_k}^{(m)}(0)| \end{aligned} \tag{8}$$

For $m>M^{k-1}$ consider $d_{k,nM^{k-1}+p}=d_{k,p}$ with n, $p=1, 2,\ldots$. Therefore the scaling function σ can be defined by

$$\sigma(t) = \lim_{k\to\infty} \frac{d_{k,q}}{d_{k+1,q}}, \qquad t=\frac{q}{M^{k+1}} \tag{9}$$

where $q=1, 2,..., M^k$.

The scaling factors α and α^z are associated with the greatest and smallest values of σ, respectively, and are given by the relations $\sigma(1/M) = 1/\alpha$ and $\sigma(0^+) = 1/\alpha^z$. Table I shows the values of α for $M = 3$, 4, and 5 and $z = 1.5$, 2, 3, 4, 6, 8, and 10. Observe that the qualitative behavior of $\alpha(z)$ is the same for all sequences with $M = 2$, 3, 4, and 5, i.e., it is a monotonically decreasing function of z and seems to diverge when $z \to 1$ (see ref. 6 for the $M = 2$ case). The scaling α^z is shown in Fig. 2. Observe that δ and a^z share the same qualitative behavior; both of them have a minimum near $z = 2$, seem to diverge for $z \to 1$, and in the limit $z \to \infty$ the relation $\delta \lesssim \alpha^z$ is satisfied (this relation was observed for $M = 2$ in ref. 8). Therefore, the question of whether δ has a limiting value when $z \to \infty$ is transformed into a similar question for α^z.

The function σ calculated for larger values of q does not give any further information, since $d_{k+1,q+M^k} = d_{k+1,q}$ and therefore $\sigma(t + 1/M) = \sigma(t)$. Figure 3 shows $\sigma(t)$ for $z = 1.5$, 2, and 10 for the trifurcations ($M = 3$). In every rational value of t $(0 < t < 1/M)$ there exists a jump in the function σ, but we observe that the discontinuities decrease rapidly as the binary expansion of the rational increases. In a crude

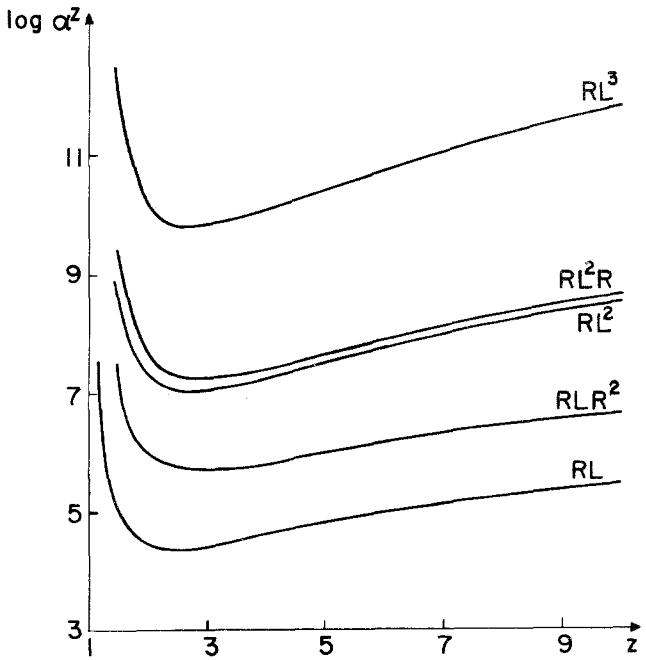

Fig. 2. Log plot of the scaling factor α^z as a function of z for the U-sequences with $M = 3$, 4, and 5.

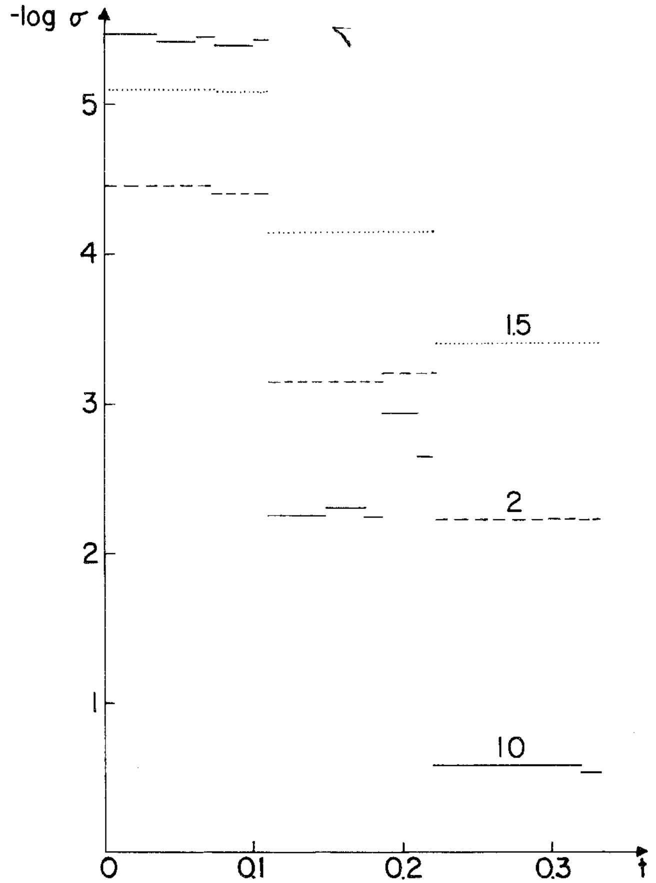

Fig. 3. Log plot of the scaling function σ for $z = 1.5$, 2, and 10 for the sequence $(RL)^{*n}$.

approximation there are M plateaus, which are divided into subplateaus. The discontinuities of the subplateaus become more and more pronounced when z increases, and they can be calculated using approximate methods (see ref. 6 for the $M = 2$ case).

4. THE FUNCTION $f(a)$

The scaling function $f(a)$ is another way to characterize the multifractal set associated with the x direction. It is more convenient than the function σ, from both theoretical and experimental points of view, since it is a smooth function.

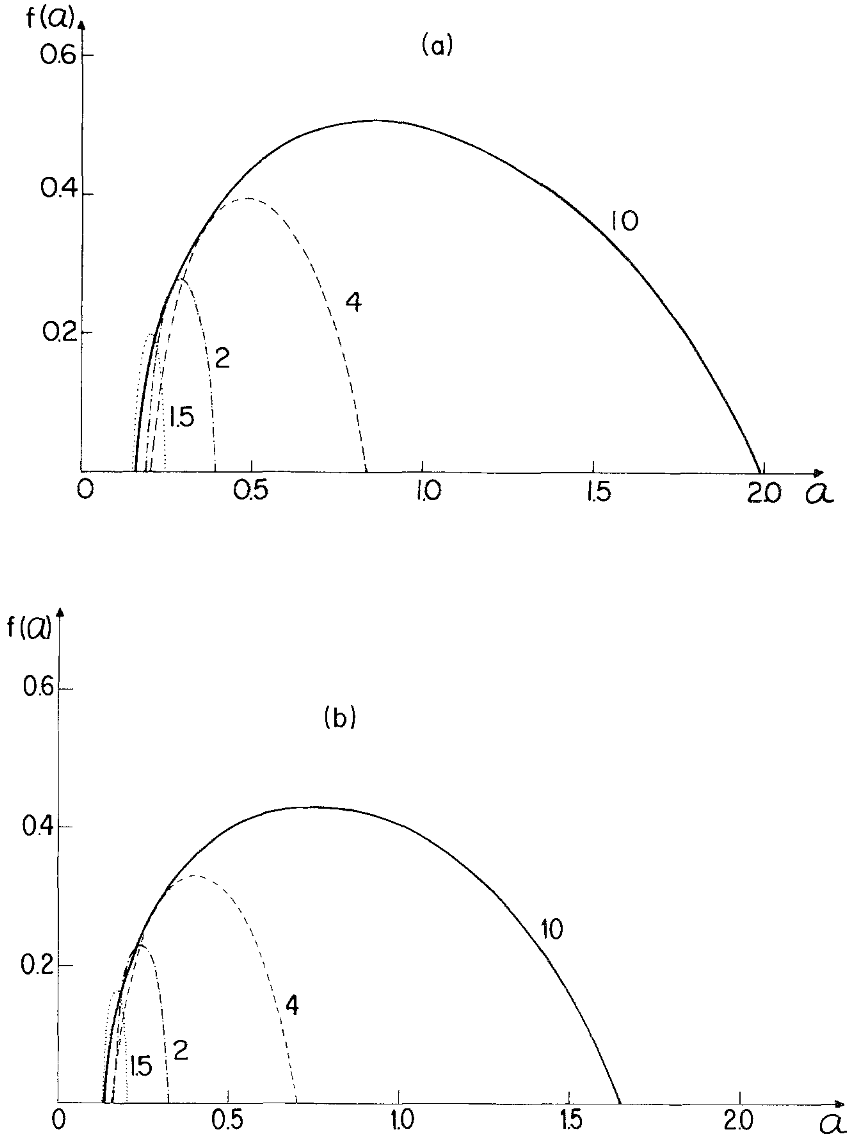

Fig. 4. The function $f(a)$ for $z = 1.5$, 2, 4, and 10 for the sequences (a) $(RL^2)^{*n}$ and (b) $(RL^3)^{*n}$.

The formalism introduced by Halsey *et al.*[18] consists in covering the attractor with boxes, indexed by i, of size l_i, and assumes that the probability density scales like $p_i \propto l_i^a$ in the limit $l_i \to 0$. The next step is to form the normalized partition function

$$\Gamma(q, \tau) = \sum_i \frac{p_i^q}{l_i^\tau} = 1 \tag{10}$$

The function $\tau(q)$ determines the function $f(a)$ through a Legendre transformation.

To study the multifractal set present at the attractor of the M-furcations, I have chosen $p_i \equiv p = 1/M^{k-1}$ for the M^k-cycle. Therefore the partition function becomes

$$\Gamma_k = \left(\frac{1}{M^{k-1}}\right)^q \sum_{m=1}^{M^{k-1}} d_{k,m}^{-\tau} \tag{11}$$

where $d_{k,m}$ is given by Eq. (8). The function $f_k(a)$ obtained by Eq. (11) converges, for k large enough, to the universal function $f(a)$. The minimal and maximal values of a, which respectively characterize the most concentrated

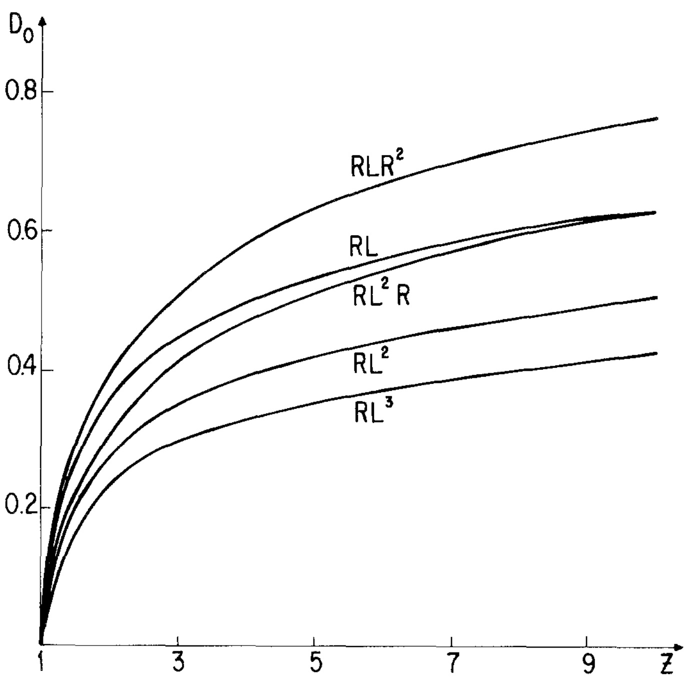

Fig. 5. Hausdorff dimension for the U-sequences with $M = 3$, 4, and 5.

and most rarefied regions of the attractor, are given by $a_{\min} = \ln M/\ln \alpha^z$ and $a_{\max} = \ln M/\ln \alpha$. Consequently $a_{\max} = z a_{\min}$ for all kinds of sequences; this is a useful relation, as it determines the exponent z associated with the maximum of the map in physical experiments.

Figure 4 shows the function $f(a)$ for $z = 1.5, 2, 4$, and 10 and for the sequences $(\mathrm{RL}^2)^{*n}$ and $(\mathrm{RL}^3)^{*n}$. In the limit $z \to 1$ the curve $f(a)$ reduces to a sharp peak at $a = 0$, since $a_{\min} = a_{\max} = 0$, and the Hausdorff dimension D_0 [which coincides with the maximum of $f(\alpha)$] goes to zero. For increasing z, D_0 increases monotonically and converges to 1 in the limit $z \to \infty$ (see Fig. 5). The behaviors of $a_{\min}$ and $a_{\max}$ for increasing z are directly related to the behaviors of α^z and α, respectively. Therefore $a_{\min}$ first grows until it reaches a maximum near $z = 2$ and then decreases, whereas $a_{\max}$ is a monotonically increasing function of z and goes to infinity in the limit $z \to \infty$.

5. CONCLUSIONS

I have studied numerically the scaling factors associated with the M-furcations ($M = 3, 4$, and 5) for single-hump one-dimensional maps given by $x' = 1 - \mu |x|^z$. The numerical data were obtained by observing the level-by-level convergence of the scalings in the M-furcation tree. When z is varied, the factors δ and α^z have similar qualitative behavious for $M = 3, 4$, and 5, i.e., they diverge for $z \to 1$ and have a minimum near $z = 2$. In the limit $z \to \infty$ I verified that $\delta \lesssim \alpha^z$, but whether these scalings diverge in this limit is a question to be worked out. The scaling α is a monotonically decreasing function of z for all sequences studied. I have also calculated the functions σ and $f(a)$ related to the multifractal set present at the accumulation points of the M-furcations.

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